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1984 J. Phys. A: Math. Gen. 17 L473

(http://iopscience.iop.org/0305-4470/17/9/004)

J. Phys. A: Math. Gen. 17 (1984) L473–L477. Printed in Great Britain

LETTER TO THE EDITOR

Inversion relation and phase transition of the Potts model

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Received 28 March 1984

Abstract. The impact of some recently discovered automorphic properties of the partition function of the Potts model on the location of the phase transition is examined, in particular in the regime with mixed ferro- and antiferomagnetic interactions.

An anisotropic q-component Potts model on a square lattice is defined by two interaction parameters K_1, K_2 in the horizontal and vertical directions. The partition function per site $Z(e^{K_1}, e^{K_2})$ is known to obey three basic functional relations (Jaekel and Maillard 1982, Maillard and Rammal 1983) generated by:
Duality: D

$$Z(e^{K_1}, e^{K_2}) = \frac{(e^{K_1}-1)}{\sqrt{q}}\frac{(e^{K_2}-1)}{\sqrt{q}} Z\left(\frac{e^{K_2}+q-1}{e^{K_2}-1}, \frac{e^{K_1}+q-1}{e^{K_1}-1}\right);$$

Inverse transform: I

$$Z(e^{K_1}, e^{K_2})Z(e^{-K_1}, 2-q-e^{K_2}) = (e^{K_2}-1)(1-q-e^{K_2});$$

Lattice symmetry: S

$$Z(e^{K_1}, e^{K_2}) = Z(e^{K_2}, e^{K_1}).$$

Additional functional relations can be obtained from the composition of these basdic involutive transforms D, I, S. We can therefore compute the value of Z at a large number of points (possibly infinite) once the value of Z is known at some point of the plane (e^{K_1}, e^{K_2}). This shows that a critical point whenever it exists is mapped into a set of critical points. A critical curve is thus mapped globally into itself by all the elements of the group of transformations generated by D, I, S. We can already verify that this necessary condition is fulfilled by the two critical curves known for the pure ferromagnetic regime ($e^{K_1} > 1$ and $e^{K_2} > 1$) and the pure antiferromagnetic regime ($e^{K_1} < 1$ and $e^{K_2} < 1$) recently found by Baxter (1982). In particular, the action of the group on a presumed critical curve, although it may lead to curves in non-physical regions of e^{K_1} and e^{K_2}, should not generate new lines in known physical regions, in any case.

For the Potts model in the mixed regime ($e^{K_1} < 1$ and $e^{K_2} > 1$ or $e^{K_1} > 1$ and $e^{K_2} < 1$) Kinzel *et al* (1981) have proposed the following equations for the critical lines:

$$(e^{K_1}+1)(1-e^{K_2}) = q, \qquad (e^{K_2}+1)(1-e^{K_1}) = q.$$

The first and second curves are respectively invariant under D, I and D, SIS. However

0305-4470/84/090473 +05$02.25

for $q=3$ the action of $(SI)^2$ on the first one generates the new critical line $e^{K_1}(1-e^{K_2}) = 3e^{K_2}$ which is not acceptable since it is located again in the region $0<e^{K_2}<1$ and $e^{K_1}>0$.

In the following we propose a systematic use of the full group G generated by D, I, S to locate globally invariant curves which may appropriately describe the transition of the Potts model in the mixed regime found by Kinzel *et al.*

To simplify the discussion we shall make use of the rationalised variables introduced by Jaekel and Maillard:

$$x=(e^{K_1}-q_+)/(e^{K_1}-q_-), \qquad y=(e^{K_2}-q_+)/(e^{K_2}-q_-), \tag{1}$$

where $q_\pm=(1-\frac{1}{2}q)\pm[\frac{1}{4}q(q-4)]^{1/2}$. The generators of the group G are now expressed as point transformations in the plane (x, y):

$$D: (x,y)\to(-q_+/y, -q_+/x), \qquad I: (x,y)\to(1/x, q_+^2/y), \qquad S: (x,y)\to(y,x).$$

As can be easily checked, D commutes with any element built from I and S. These elements are themselves divided into two classes: those exchanging x and y and those operating separately on x and y. In the latter class are elements containing always an even number of S-operations such as, for $m=0, 1, \ldots, \infty$,

$$(SI)^{2m}: (x,y)\to(q_+^{2m}x, q_+^{-2m}y), \qquad (IS)^{2m}: (x,y)\to(q_+^{-2m}x, q_+^{2m}y),$$

$$I(SI)^{2m}: (x,y)\to(1/q_+^{2n}x, q_+^{2m+2}/y), \qquad SIS(IS)^{2m}: (x,y)\to(q_+^{2m+2}/x, 1/q_+^{2m}y).$$

At criticality the difference between the formulations on the lattice and on the dual latitice is vanishing because order and disorder are crossing into each other; it would be more appropriate to locate the critical curves whenever they are in a physical range in the plane (e^{K_1}, e^{K_2}) as invariant curves under the '*extended*' *dualities*: $D(SI)^{2m}$, $D(IS)^{2m}$, $DI(SI)^2$, $D(SIS)(IS)^{2m}$, for all integer m.

The first transform $D(SI)^{2m}$ is the mapping

$$(x,y)\to(-q_+^{2m+1}/y, -1/q_+^{2m-1}x),$$

and admits possibly the invariant curve determined by the two equations

$$xy=-q_+^{2m+1} \qquad \text{and} \qquad xy=-q_+^{-2m+1}.$$

The curve is then non-trivial if $q_+^{2m}=\pm1$ or $q_+^{4m}=1$. This is equivalent to saying that $q_+=\exp(2i\pi/4m)^k$ or because of (1):

$$q=2-2\cos(2\pi k/4m)=2+2\cos[2\pi(2m+k)/4m].$$

Hence q is a Tutte–Beraha number $q_{4m,2m+k}$. We obtain thus two curves $xy=\pm q_+$; they correspond exactly to the ferro- and antiferromagnetic critical curves. Observing that any number $0<q<4$ may be arbitrarily close to a $q_{4m,2m+k}$ for an appropriate choice of m and k, we conclude that the result holds for any q and can be eventually extended to $q>4$ by analytic continuation. Finally, had we started from $D(IS)^m$ we would obtain precisely the same result.

To deal with the Potts model in the mixed regime we consider the two remaining 'extended' dualities:

$$DI(SI)^{2m}: (x,y)\to(-q_+^{-2m-1}y, -q_+^{2m+1}x),$$

$$DSIS(IS)^{2m}: (x,y)\to(-q_+^{2m+1}, -q_+^{-2m-1}x),$$

which admit the invariant curves

$$y=-q_+^{2m+1}x \qquad \text{and} \qquad x=-q_+^{2m+1}y.$$

Obviously the curves are mapped into each other by S. Hence we shall designate a generic curve (Γ_m) by

$$y=-q_+^{2m+1}x$$

and let m take values in both positive and negative integer values. (Γ_m) has now the equation

$$e^{K_1+K_2}-e^{K_1}q_+\frac{1+q_+^{2m-1}}{1+q_+^{2m-1}}-e^{K_2}\frac{1}{q_+}\frac{1+q_+^{2m-2}}{1+q_+^{2m-1}}+1=0 \qquad (2)$$

which is obviously invariant under the action of D; S maps (Γ_m) into (Γ_{-m-1}). So that the totality of all the curves (Γ_m) $m\in\mathbb{Z}$ is globally invariant under G.

Let us consider some simple cases.

For $q=2$ we have $q_+=1$ and there are only two curves corresponding to $m=0$ and $m=-1$; they are precisely those of the Ising model with mixed interactions.

For $q=3$ we have $q_+=j=e^{2i\pi/3}$ and three curves of equations:

$$e^{K_1+K_2}-e^{K_1}+2e^{K_2}+1=0, \qquad e^{K_1+K_2}+\tfrac{1}{2}(e^{K_1}+e^{K_2})+1=0,$$

$$e^{K_1+K_2}+2e^{K_1}-e^{K_2}+1=0.$$

The middle one is unphysical because it has no branch in the physical quadrant whereas the other two are precisely located in the mixed regime of the three-state Potts model. They depart from both axes at (0, 1) and (1, 0) as for $q=2$, and not at (0, 2) and (2, 0) as in the conjecture of Kinzel *et al* (see figure 1).

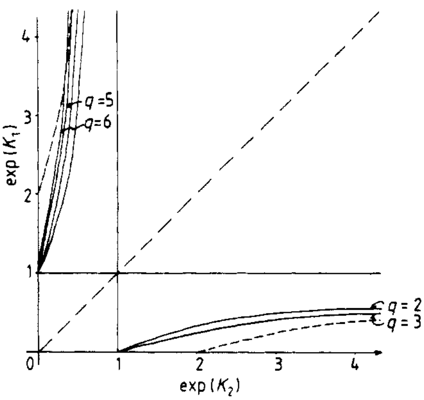

Figure 1. Curves (Γ_0) and (Γ_1) for $q=2$, 3, 5, 6. The Kinzel *et al* curves are traced with dotted lines.

Consider now $q=5$ and 6, for which $q_+=\frac{1}{2}(-3+\sqrt{5})$, $(-2,+\sqrt{3})$ is real and negative. We have an infinite number of curves now; however, *only two are in the physical area of the first quadrant* corresponding to $m=0$ and $m=-1$. This can be seen upon inspection of (2) and by recalling that q_+ is small and negative; so that for m large enough the (Γ_m) recedes in the unphysical area. In fact for q large we have $q_+\simeq(-1/q)$ and we reach again the same conclusion.

Suppose now that $q = e^{(2i\pi k/N)}$ or q is a Tutte–Beraha number:

$$q = q_{N,k} = 2 + 2\cos(2\pi k/N);$$

then the corresponding curves (Γ_m) take the form

$$e^{K_1+K_2} - e^{K}\cos[\pi k(2m-1)/N/\cos[\pi k(2m+1)/N] - e^{K}\cos[\pi k(2m+3)/N/\cos[\pi k(2m+1)/N] + 1 = 0.$$

One might wonder what would happen to the Potts model with $q = q_{5,1} = 2 + 2\cos(2\pi/5)$ equivalent to the critical hard-hexagon model in the mixed regime. As pointed out, the group generated by I and S is finite and of order 10. Inspection shows that we do have ten curves; however, only two again corresponding to $m = 0$, 4 are located in the physical region, and are mapped into each other by S.

For $q = 4$, the limit between real and unimodular q_+, care should be taken for example by considering a $q_{N,k}$ arbitrarily close to 4, then taking the limit $q \to 4$.

Thus we have shown that by considering a group G which generates automorphic functional relations for the partition function per site of the Potts model we are led to consider 'extended' dualities, which yields all the known critical lines and a set of plausible lines in the mixed regime. It is very tempting to identify thus (Γ_0) (invariant under D and I) as well as (Γ_{-1}) (invariant under D and SIS, symmetrised inversion) with the actual transition lines, for they coincide with the exactly known results for $q = 2$ and are supported in the case $q = 3$ by the numerical findings of Kinzel *et al* on top of their invariance properties†. Of course we cannot say anything about the nature of the transition nor can we calculate the partition function on these lines, although it depends only on one variable. However, let

$$Z_{2m+1}(x) = Z(x, y)|_{y=-q_+^{2m+1}x};$$

then we have the following functional relations for all m:

$$Z_1(x)Z_1\left(\frac{1}{x}\right) = \varphi_I(x) = -qq_+\frac{1-x/q_+}{1-x}\frac{1-1/xq_+}{1-1/x},$$

$$Z_{2m+1}(x) = Z_{-2m-1}(-q_+^{2m+1}x), \qquad Z_{2m+1}(x) = Z_{2m-3}(q_+^2x)\varphi_I(x)\varphi_I^{-1}(-1/q_x^{2m-1}x).$$

Unlike the critical ferro- or antiferromagnetic case for $q > 4$ where $(SI)^2$ performs an iteration along the critical curve, here $(SI)^2$ merely iterate from curve to curve and the group contains no element which iterates along the same (Γ_m). Thus the value of $Z_{2m+1}(x)$ is not sufficiently known from the previous structure.

To close we would like to point out that the subset of transformations generated by S and I containing an odd number of S presents also some interesting aspects. Consider first the transformations $(ISI)(SI)^{2m}$ which exchange x and y.

$$ISI(SI)^{2m}:(x, y) \to (y/q_+^{2m+2}, q_+^{2m+2}x).$$

† The Kinzel *et al* curve in the x, y variables has the equation

$$xy - q_+ - x\frac{q_+^{-1} - q_+ - q}{1 - q_+^{-2} - 1} - y\frac{q_+ - q_+^{-1} - q}{1 - q_+^{-2} - q} = 0.$$

It cannot be described as the set of invariant points of any element of the group G. The same can be said about its S transform.

The invariant curves are of the form

$$y = q_+^{2m+2} x.$$

They are also D invariant, as well as transforming among themselves under any combination of S and I. But as can be seen in the variables e^{K_1} and e^{K_2},

$$e^{K_1+K_2} - e^{K_1} q_+ \frac{1-q_+^{2m}}{1-q_+^{2m+2}} - e^{K_2} \frac{1}{q_+} \frac{1-q_+^{2m+4}}{1-q_+^{2m+2}} + 1 = 0, \tag{3}$$

they are located all the time in the unphysical part of (e^{K_1}, e^{K_2}). For example $m = 0$ yields

$$e^{K_1+K_2} + 1 - e^{K_2} q_+^{-1}(1 + q_+^2) = 0,$$

$$q = 2, \qquad q_+ = i, \qquad e^{K_1+K_2} = 0,$$

$$q = 3, \qquad q_+ = j, \qquad 1 + e^{K_1+K_2} + e^{K_2} = 0,$$

$$q = 5, \qquad q_+ = \frac{(-3+\sqrt{5})}{2}, \qquad 1 + e^{K_1+K_2} - e^{K_2} \frac{1 + [\frac{1}{2}(\sqrt{5}-3)]^2}{\frac{1}{2}(\sqrt{5}-3)} = 0.$$

From (3) it is clear that for $q \geqslant 4$, q is negative and for all $m > 0$ the curves are unphysical. For $q = 2$ there exist only two curves: the one found above and $e^{K_1} = e^{K_2}$ when m is odd. Similarly for $q = 3$ besides the curve found for $m = 0$ we have two more:

$$e^{K_1+K_2} + e^{K_1} + 1 = 0 \qquad \text{and} \qquad e^{K_1} = e^{K_2}.$$

But we recall that these transforms are not dualities, hence the bisector $e^{K_1} = e^{K_2}$ cannot be a critical line.

To conclude it would be of interest to see how these ideas can be extended to anisotropic $Z(N)$-spin systems, to different types of lattices, and a check by numerical simulation for values of q different from 2 and 3 is most welcome.

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